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Note

Numerical Inversion of Mellin and Two-Sided Laplace Transforms

The Mellin transform has many uses in physics and applied mathematics, in particular, in the solution of problems of elasticity [1, 2], and in the theory of cosmic ray showers [3–6]. Recently Tsamasphyros and Theocaris [7] have presented a method for numerical inversion of Mellin transforms by expanding the inverse transforms in terms of Laguerre polynomials which was extended by Tsamasphyros and Chrysakis [8]. In this note we use the identity

$$F(s) = \int_0^\infty x^{s-1} f(x)\, dx = \int_{-\infty}^\infty e^{-ts} f(e^{-t})\, dt \tag{1}$$

to express the Mellin transform in terms of a two-sided Laplace transform. We then show that a slight generalization of an algorithm developed by Dubner and Abate [9] and extended by Crump [10] for numerical inversion of the one-sided Laplace transform, can be applied successfully to numerical inversion of the two-sided Laplace transform.

A derivation parallel to that given by Dubner and Abate suffices to show that $f(e^{-t})$ can be expressed in the exact form

$$f(e^{-t}) = f_a(e^{-t}) + E_+ + E_- \tag{2}$$

for $-T \leqslant t \leqslant T$, where

$$\begin{aligned} f_a(e^{-t}) = \frac{e^{at}}{2T} \Big[F(a) + 2 \sum_{r=1}^\infty \Big\{ \operatorname{Re} F\left(a + \frac{\pi i r}{T}\right) \cos\left(\frac{\pi r t}{T}\right) \\ - \operatorname{Im} F\left(a + \frac{\pi i r}{T}\right) \sin\left(\frac{\pi r t}{T}\right) \Big\} \Big] \end{aligned} \tag{3}$$

is the approximation actually used, and E_+ and E_- are error terms given by

$$\begin{aligned} E_+ &= \sum_{n=1}^\infty e^{-2naT} f(e^{-2nT-t}), \\ E_- &= \sum_{n=1}^\infty e^{2naT} f(e^{2nT-t}). \end{aligned} \tag{4}$$

The parameter a is arbitrary, restricted only by the fact that the Mellin transform converges in a strip $\alpha < \operatorname{Re} s < \beta$ [11], where α and β can equal $-\infty$ or $+\infty$, respectively.

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It is obvious that the choice of a and T both influence the accuracy and rate of convergence of Eq. (3). Several considerations are needed in the choice of these parameters. If t is held fixed then as T increases the rate of convergence of the sum in Eq. (3) decreases. This suggests choosing T as small as possible. If T is chosen too small then the errors in Eq. (4) increase. A good rule of thumb for choosing T is to let it be no more than twice the maximum value of t, i.e., choose T so that $|t/T| \leqslant \frac{1}{2}$. Sometimes convergence is too show even with this choice of T. It then may be advantageous to set $t = T$ and to use an acceleration method, of which the epsilon algorithm is a good example [10], although not the only possibility [12, 13]. When the strip of convergence of $F(s)$ has a finite width (α, β) then a study of several examples suggests that a should be chosen to be

$$a = \tfrac{1}{2}(\alpha + \beta). \tag{5}$$

We have used the inversion formula in Eq. (4) on several examples, some of them also considered by Tsamasphyros and Theocaris. These are

1. $f_1(x) = \exp(-2x)$, $\quad F_1(s) = 2^{-s}\Gamma(s)$,
2. $f_2(x) = (1 - x)/(1 - x^{20})$, $\quad F_2(s) = \pi \sin\left(\frac{\pi}{20}\right)\Big/\left[20 \sin\left(\frac{\pi s}{20}\right) \sin\left(\frac{\pi(s+1)}{20}\right)\right]$,
3. $f_3(x) = \cos(3x)\exp(-4x)$, $\quad F_3(s) = 5^{-s}\Gamma(s)\cos(s \cdot \tan^{-1}(\frac{3}{4}))$.

A tabulation of the relative errors, the number of terms required, and the values of T chosen in the series in Eq. (3) is given in Table I.[1] The function $F_2(s)$ required a large number of terms for the inversion with $T = 10$, as shown, but convergence with approximately one-fourth of the number of terms was achieved by using $T = 2.5$, with considerable degradation in accuracy at $x = 0.1$ and with some degradation in accuracy for $x \geqslant 6$. We have also inverted the transform pair $f_4(x) = x^{2.5}[1 - H(x - 1)]$, $F_4(s) = 1/(s + 2.5)$, where $H(x)$ is the Heaviside function, finding some difficulty with Gibbs' phenomenon at $x = 1$, as in [7], while the accuracy elsewhere is similar to the accuracy in Table I.

We conclude that the present technique is stable over a useable range in x and is simple to program, provided that $F(s)$ can be evaluated for complex s. Furthermore, it does not require high-order differences as in the Laguerre polynomail technique suggested by Tsamasphyros and Theocaris, although their method has the advantage of requiring $F(s)$ for real s only. There are indications that there may be some advantages in accuracy in using the present method. As an example we have compared the relative errors obtained using the best method reported by Tsamasphyros and Theocaris [7] with those in Table I for the inversion of $F_2(s)$. The relative errors by the Laguerre inversion technique are: at $x = 0.1$, $R = 2.0 \times 10^{-3}$ (compared with -7×10^{-8} from Table I), at $x = 0.5$, $R = 9 \times 10^{-7}$ (2×10^{-8}), at $x = 1.0$,

[1] Note that the errors in [7] are absolute errors, while our errors are relative errors, defined by $R_i = (f_i(x)_{\text{appr.}} - f_i(x)_{\text{exact}})/f_i(x)_{\text{exact}}$, where $f_i(x)_{\text{appr.}}$ is the result of using Eq. (3) with N_i terms.

$R = -4 \times 10^{-4}$ (3×10^{-9}), at $x = 3.0$, $R = 3 \times 10^{5}$ (1×10^{-8}). A comparison of relative errors for $F_3(s)$ shows that the relative errors of the Laguerre inversion technique are: At $x = 0.1$, $R = -2 \times 10^{-7}$ (1×10^{-8}), at $x = 0.5$, $R = 7 \times 10^{-6}$ (4×10^{-8}), at $x = 1.0$, $R = 2 \times 10^{-6}$ (-5×10^{-9}), at $x = 3.0$, $R = 2 \times 10^{-5}$ (3×10^{-6}), at $x = 3.0$, $R = 2 \times 10^{-5}$ (3×10^{-6}), at $x = 5.0$, $R = 2 \times 10^{-4}$

TABLE I

Relative Errors, R_i, and Number of Terms, N_i, for the Numerical Inversion of $F_i(s)$, $i = 1, 2, 3$.[a]

	$a = 2$		$a = 10$		$a = 2$	
x	R_1	N_1	R_2	N_2	R_3	N_3
0.1	5(−8)	101	−7(−8)	394	1(−8)	86
0.2	3(−9)	98	3(−9)	321	−1(−8)	86
0.3	5(−9)	98	−7(−8)	254	−7(−9)	87
0.4	5(−9)	100	−1(−8)	246	1(−9)	81
0.5	4(−9)	94	2(−8)	224	4(−8)	93
0.6	6(−9)	99	5(−9)	224	−2(−8)	82
0.7	5.(−9)	90	−1(−7)	182	−1(−8)	83
0.8	3(−9)	95	−6(−9)	198	−8(−9)	84
0.9	−7(−9)	91	2(−9)	206	−8(−9)	84
1.0	−4(−9)	95	3(−9)	206	−5(−9)	83
2.0	1(−9)	99	−4(−8)	211	−2(−7)	79
3.0	1(−8)	100	1(−8)	252	3(−6)	98
4.0	4(−8)	110	1(−8)	272	3(−5)	114
5.0	1(−7)	117	2(−9)	309	−8(−4)	127
6.0	3(−7)	121	7(−9)	311	1(−1)	136
7.0	−2(−7)	120	−2(−9)	333	−7(0)	157
8.0	−9(−6)	143	3(−9)	345		
9.0	−8(−5)	149	2(−9)	354		
10.0	−6(−4)	150				

[a] $T_1 = 20$, $T_2 = T_3 = 10$.

(-8×10^{-4}), at $x = 7.0$, $R = 6$ (-7). It is not profitable to make a more detailed comparison without taking into account the relative running times of the respective inversions.

Note. All of the calculations were run in double precision on an IBM 360/95 computer and checked for accuracy on an Amdahl 470 V/6 computer.

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